Stochastic Processes and their Applications 11 (1981) 79–89
North-Holland Publishing Company

AN APPLICATION OF A MARTINGALE CENTRAL LIMIT THEOREM TO THE STANDARD EPIDEMIC MODEL

Ray WATSON
Department of Statistics, University of Melbourne, Parkville, Australia

Received 14 June, 1979
Revised 10 October, 1979

In this paper, useful limit results for the standard epidemic model are proved using a martingale central limit theorem given by McLeish [2]. These results do not depend on the solutions of the deterministic equations, and also cover the case of a small initial number of infectives. Further, the results find immediate application in approximating the size distribution and in estimating the threshold parameter.

Martingale central limit theorem	estimation
epidemic process	size distribution

1. Introduction and definitions

The standard epidemic process is a continuous time Markov process $\{\boldsymbol{P}(t), t \geq 0\}$, on the state space $\{(x, y, z): x, y, z$ non-negative integers and $x+y+z=n+a\}$, with initial conditions $\boldsymbol{P}(0)=(n, a, 0)$ and transition probabilities given by

$$\mathbf{P}[\boldsymbol{P}(t+\mathrm{d}t)=(x-1, y+1, z) \mid \boldsymbol{P}(t)=(x, y, z)]=n^{-1}\beta xy\,\mathrm{d}t+\mathrm{o}(\mathrm{d}t),$$

$$\mathbf{P}[\boldsymbol{P}(t+\mathrm{d}t)=(x, y-1, z+1) \mid \boldsymbol{P}(t)=(x, y, z)]=\gamma y\,\mathrm{d}t+\mathrm{o}(\mathrm{d}t).$$

The components of $\boldsymbol{P}(t)$ represent the numbers of susceptibles, infectives and removed cases respectively, at time t, and we let $\boldsymbol{P}(t)=(X(t), Y(t), Z(t))$. The parameter $\theta=\beta/\gamma$ is called the threshold parameter and its value greatly influences the behaviour of the process.

Let $\mathcal{P}_n=\{\boldsymbol{P}_{nj}, j=0,1,2,\ldots\}$ where $\boldsymbol{P}_{nj}=(X_{nj}, Y_{nj}, Z_{nj})$, be defined by initial conditions

$$\boldsymbol{P}_{n0}=(X_{n0}, Y_{n0}, Z_{n0})=(n, a_n, 0),$$

and transition probabilities

$$\mathbf{P}[\boldsymbol{P}_{n,j+1}=(x-1, y+1, z) \mid \boldsymbol{P}_{nj}=(x, y, z)]=\lambda_n x(1+\lambda_n x)^{-1},$$

$$\mathbf{P}[\boldsymbol{P}_{n,j+1}=(x, y-1, z+1) \mid \boldsymbol{P}_{nj}=(x, y, z)]=(1+\lambda_n x)^{-1},$$

where λ_n denotes a positive constant; with the provision that transitions cease at a Markov stopping time N_n.

Let $J_n(A)$ denote the number of transitions until the process $\mathscr{P}_n$ reaches the set of states A, i.e.,

$$J_n(A) = \min\{j: \boldsymbol{P}_{nj} \in A\}.$$

We stipulate that the stopping time N_n must be no greater than $J_n(A_0)$, where $A_0 = \{(\xi, \eta, \zeta): \xi \leq 0\}$. We note that $J_n(A_0) = \sum_{x=1}^{n} j_n(x)$, where $j_n(x)$ denotes the number of transitions while $X_{nj} = x$. Now $j_n(x)$, $x = 1, 2, \ldots, n$ are independent geometric random variables, and so $J_n(A_0)$ is a proper random variable for each $n = 1, 2, \ldots$. It follows that

$$\mathbf{P}(N_n < \infty) = 1.$$

We let $\boldsymbol{P}_n = (X_n, Y_n, Z_n)$ denote the final state of the process, so that $\boldsymbol{P}_n = \lim_{j\to\infty} \boldsymbol{P}_{nj} = \boldsymbol{P}_{n,N_n}$. It will be observed that the process $\mathscr{P}_n$ is not exactly equivalent to the embedded random walk for the standard epidemic process, since transitions are possible even if the number of infectives, Y_{nj} becomes non-positive. Of course, the standard epidemic model can be recovered by a suitable choice for N_n, viz., $N_n = J_n(B_0)$, where $B_0 = \{(\xi, \eta, \zeta): \eta \leq 0\}$. The situation is illustrated in Fig. 1.

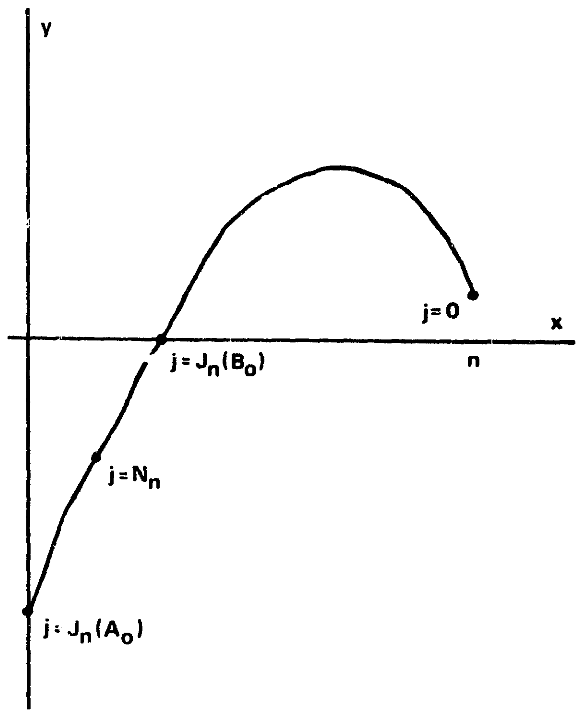

Fig. 1. Realization of a major outbreak in the model $\mathscr{P}_n$, with possible stopping times indicated.

We define the derived process $\mathscr{K}_n = \{K_{nj}, j = 0, 1, 2, \ldots\}$, where

$$K_{nj} = -\nabla H_1(X_{nj}) + \lambda_n Z_{nj}$$

with $H_\nu(x)=\sum_{k=1}^{x} k^{-\nu}$, $x=1,2,\ldots,$ and $H_\nu(0)=0$, $\nu=1,2,\ldots;$

$$\nabla H_\nu(x)=H_\nu(n)-H_\nu(x)=\sum_{k=x+1}^{n} k^{-\nu}.$$

It is easily verified that, for each $n=1,2,\ldots,\mathscr{K}_n$ is a zero-mean square-integrable martingale with respect to $\mathscr{F}_n=\{F_{nj}, j=0,1,2,\ldots\}$, where F_{nj} denotes the σ-field generated by the process $\mathscr{P}_n$ up to the jth transition. We define

$$K_n=\lim_{j\to\infty} K_{nj}=K_{n,N_n} \quad \text{and} \quad \Delta K_{nj}=K_{nj}-K_{n,j-1}.$$

We also define the variance estimates

$$U_{nj}^2=\sum_{i=1}^{j} (\Delta K_{ni})^2=\nabla H_2(X_{nj})+\lambda_n^2 Z_{nj}, \quad \text{with } U_n^2=\sum_{i=1}^{\infty} (\Delta K_{ni})^2;$$

$$V_{nj}^2=\sum_{i=1}^{j} \mathbf{E}[(\Delta K_{ni})^2 \mid F_{n,i-1}]=\lambda_n \sum_{i=0}^{j-1} X_{ni}^{-1}, \text{ with } V_n^2=\sum_{i=1}^{\infty} \mathbf{E}[(\Delta K_{ni})^2 \mid F_{n,i-1}];$$

and

$$\sigma_{nj}^2=\operatorname{var}(K_{nj})=\mathbf{E}(U_{nj}^2)=\mathbf{E}(V_{nj}^2), \quad \text{with } \sigma_n^2=\lim_{j\to\infty}\sigma_{nj}^2.$$

It is assumed throughout that $\lambda_n=n^{-1}\theta$, where the threshold parameter $\theta>1$. Thus we are concerned with the asymptotic behaviour, as $n\to\infty$, of the model for a given supercritical value of the threshold parameter. Other results could be derived for different forms of λ_n, but this assumption corresponds to the standard epidemic model and is the most appropriate for applications.

In Section 2 it is proved that, under specified conditions, as $n\to\infty$

$$K_n/\sigma_n \xrightarrow{\text{d}} \mathrm{N}(0,1), \qquad K_n/U_n \xrightarrow{\text{d}} \mathrm{N}(0,1) \quad \text{and} \quad K_n/V_n \xrightarrow{\text{d}} \mathrm{N}(0,1). \tag{1}$$

In Theorem 1 it is shown that (1) holds if $a_n\to\infty$, $n^{-1}a_n\to\alpha$, where $\alpha>0$, $n^{-1}N_n\leq\nu$ and $n^{-1}N_n \xrightarrow{\text{p}} \tau$ as $n\to\infty$, where ν and τ are positive constants. In Theorem 2 it is shown that (1) holds conditional on the occurrence of a major outbreak if $a_n=a$, where $a>0$, and if in the event of a major outbreak $n^{-1}N_n\leq\nu$ and $n^{-1}N_n\xrightarrow{\text{p}}\tau$ as $n\to\infty$, where ν and τ are positive constants. We note that $J_n(B_0)\leq 2n+a_n$, so that for any applications the bound $n^{-1}N_n\leq\nu$ will be satisfied. Further, in Lemma 4 it is shown that $n^{-1}J_n(B_0)$ converges in probability so that (1) applies to a completed major outbreak for the standard epidemic model.

The most useful results are derived from the asymptotic normality of K_n/U_n. The result $K_n/\sigma_n\xrightarrow{\text{d}}\mathrm{N}(0,1)$ under the conditions of Theorem 1, is equivalent to results derived previously by Nagaev and Startsev [3] and by Barbour [1], in which σ_n is replaced by an asymptotically equivalent deterministic approximation. The result

$K_n/U_n \xrightarrow{d} N(0, 1)$ represents an improvement in that the dependence on the deterministic approximation is removed and a comparatively simple expression results, viz.,

$$\frac{-n\nabla H_1(X_n)+\theta Z_n}{\sqrt{n^2\nabla H_2(X_n)+\theta^2 Z_n}} \xrightarrow{d} N(0, 1).$$

Further, Theorem 2 provides another improvement by demonstrating conditional convergence to normality in the case of a fixed initial number of infectives.

As for any limit result as $n\to\infty$, the most important application is the derived approximation for large values of n. Let $X = X_{n,\nu_n}$ and $Z = Z_{n,\nu_n}$, then from the asymptotic normality of K_n/U_n we deduce that, if n and ν_n are large, we have

$$\frac{-n\nabla H_1(X)+\theta Z}{\sqrt{n^2\nabla H_2(X)+\theta^2 Z}} \overset{d}{\simeq} N(0, 1). \tag{2}$$

We observe that (2) is effectively a time independent result applying provided only that there has been a large number of transitions – which, in terms of the standard epidemic model means the occurrence of a major outbreak. Further, Lemma 4 indicates that the result (2) also holds at the end of a major outbreak, so that we have

$$\frac{-n\nabla H_1(n-S_n^+)+\theta(S_n^+ + a_n)}{\sqrt{n^2\nabla H_2(n-S_n^+)+\theta^2(S_n^+ + a_n)}} \xrightarrow{d} N(0, 1), \tag{3}$$

where S_n^+ denotes the size of a major outbreak. This result, in association with the branching process approximation for the size of a minor outbreak and the probability of a minor outbreak, provides a simple and quite good approximation for the size distribution for moderate population sizes. This is illustrated in Fig. 2, where the approximation derived using (3) is compared with the exact size distribution.

Further, as has been indicated by Watson [4, 5], the result also finds application in estimation of θ. The estimator

$$\bar{\theta} = n\nabla H_1(X)/Z$$

suggested by (2), is such that

$$\frac{(\bar{\theta}-\theta)Z}{\sqrt{n^2\nabla H_2(X)+\theta^2 Z}} \xrightarrow{d} N(0, 1)$$

enabling approximate inference on θ for moderate population sizes.

We note that all the above results apply equally well to the continuous time epidemic process for which the process $\mathcal{P}_n$, stopped at time $J_n(B_0)$, is the embedded random walk.

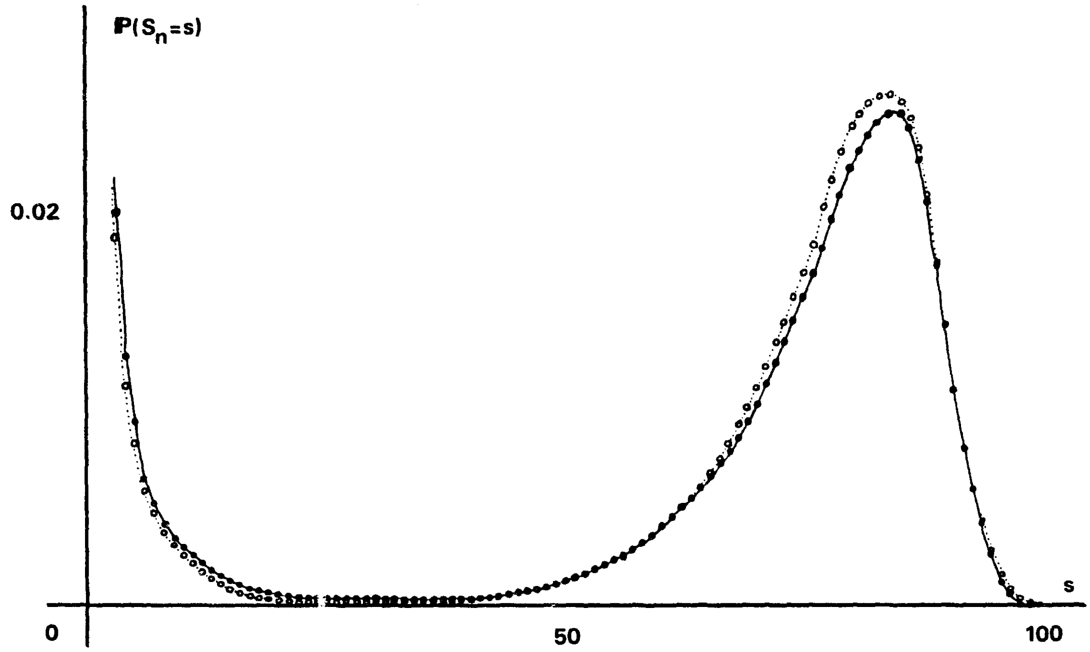

Fig. 2. Comparison of the approximation to the size distribution based on Eq. (3), with the exact size distribution, for $n = 100$, $\theta = 2$; $\cdots\circ\cdots\circ\cdots\circ\cdots$ approximation, —•—•— exact.

2. Results

Lemma 1. *For the sequence of processes $\mathscr{P}_n$, $n = 1, 2, \ldots$, defined above, if $a_n \to \infty$, $n^{-1}a_n \to \alpha$, where $\alpha \leq 0$, and if $\lambda_n = n^{-1}\theta$, where $\theta > 1$, then for any $\phi > 0$, as $n \to \infty$*

$$n^{-1}X_{n,[n\phi]} \xrightarrow{L^2} u(\phi), \tag{4a}$$

$$n^{-1}Y_{n,[n\phi]} \xrightarrow{L^2} v(\phi), \tag{4b}$$

$$n^{-1}Z_{n,[n\phi]} \xrightarrow{L^2} w(\phi), \tag{4c}$$

where u, v and w are continuous functions.

Proof. Let $A_x = \{(\xi, \eta, \zeta): \xi \leq x\}$, so that $J_n(A_x)$ denotes the number of transitions until $X_{nj} = x$. We show that, for $0 < u < 1$, as $n \to \infty$

$$n^{-1}J_n(A_{[nu]}) \xrightarrow{L^2} \phi(u)$$

and hence that (4a) holds.

We have

$$J_n(A_x) = \sum_{\xi=x+1}^{n} j_n(\xi),$$

where $j_n(\xi)$ denotes the number of transitions while $X_{nj} = \xi$. Now, $j_n(\xi), \xi = 1, 2, \ldots, n$, are independent geometric random variables, and

$$\mathbf{E}\{j_n(\xi)\} = 1 + \lambda_n^{-1}\xi^{-1}, \qquad \operatorname{var}\{j_n(\xi)\} = \lambda_n^{-1}\xi^{-1} + \lambda_n^{-2}\xi^{-2}.$$

Hence we have

$$\mathbf{E}\{J_n(A_x)\} = n - x + \lambda_n^{-1}\nabla H_1(x), \quad \operatorname{var}\{J_n(A_x)\} = \lambda_n^{-1}\nabla H_1(x) + \lambda_n^{-2}\nabla H_2(x),$$

from which we deduce that, as $n \to \infty$, with $\lambda_n = n^{-1}\theta$,

$$n^{-1}J_n(A_{[nu]}) \xrightarrow{L2} \phi(u) = 1 - u - \theta^{-1}\ln u.$$

Now, let $R_n(u)$ be such that

$$J_n(A_{[nu]}) + R_n(u) = [n\phi(u)]$$

so that, as $n \to \infty$

$$n^{-1}R_n(u) \xrightarrow{L2} 0.$$

If we let

$$\Delta X_{n,R_n} = X_{n,[n\phi]} - X_{n,J(A_{[nu]})}$$

so that $\Delta X_{n,R_n}$ denotes the change in X_{nj} in R_n transitions, and $|\Delta X_{n,R_n}| \leq R_n$, then, as $n \to \infty$,

$$n^{-1}\Delta X_{n,R_n} \xrightarrow{L2} 0$$

and therefore, since

$$X_{n,[n\phi]} = [nu] + \Delta X_{n,R_n}$$

it follows that, for $\phi > 0$, as $n \to \infty$

$$n^{-1}X_{n,[n\phi]} \xrightarrow{L2} u(\phi),$$

where $u(\phi)$ is the solution of $1 - u - \theta^{-1}\ln u = \phi$ in the interval $0 < u < 1$.

An exactly similar method can be used to show (4c) and then (4b) follows since $X_{nj} + Y_{nj} + Z_{nj} = n + a_n$.

Lemma 2. *If $\lambda_n = n^{-1}\theta$ and if $n^{-1}N_n \leq \nu$, then* $\mathbf{E}[n(X_n + 1)^{-2}] \to 0$.

Proof. We have

$$\mathbf{P}(X_n \leq [n\varepsilon]) \leq \mathbf{P}(J_n(A_{[n\varepsilon]}) \leq n\nu)$$

with $J_n(A_{[n\varepsilon]})=\sum_{\xi=[n\varepsilon]+1}^{n} j_n(\xi)$, where $j_n(\xi)$, $\xi=1,2,\ldots,n$ are as defined in Lemma 1. Thus, for any $\nu>0$, we can choose $\varepsilon>0$ so that $\mathbf{E}[J_n(A_{[n\varepsilon]})]>n\nu$. Further, with $\lambda_n=n^{-1}\theta$, it is easily shown that the fourth central moment of $J_n(A_{[n\varepsilon]})$ is such that

$$\mu_4[J_n(A_{[n\varepsilon]})]\sim c_4(\varepsilon)n^2 \quad \text{as } n\to\infty,$$

where $c_4(\varepsilon)$ is a positive constant. So, using the fourth moment Chebyshev inequality, we obtain

$$\mathbf{P}(X_n\leq[n\varepsilon])\leq\mathbf{P}(J_n(A_{[n\varepsilon]})\leq n\nu)\leq cn^{-2}.$$

Therefore,

$$\begin{aligned}\mathbf{E}[n(X_n+1)^{-2}]&=\sum_{x=0}^{[n\varepsilon]} n(x+1)^{-2}p(x)+\sum_{x=[n\varepsilon]+1}^{n} n(x+1)^{-2}p(x)\\&\leq cn^{-1}+\varepsilon^{-2}n^{-1}.\end{aligned}$$

Hence the result.

Theorem 1. *For the sequence of processes $\mathcal{P}_n$, $n=1,2,\ldots$, defined above, if $a_n\to\infty$, $n^{-1}a_n\to\alpha$, where $\alpha\geq 0$, and $\lambda_n=n^{-1}\theta$, where $\theta>1$, then*

$$K_n/\sigma_n \xrightarrow{\text{d}} \mathrm{N}(0,1) \quad \text{as } n\to\infty,$$

provided that the stopping time N_n is such that $n^{-1}N_n\leq\nu$ and $n^{-1}N_n\xrightarrow{\text{p}}\tau$, where ν and τ are positive constants. Further, $U_n/\sigma_n\xrightarrow{\text{p}}1$ and $V_n/\sigma_n\xrightarrow{\text{p}}1$ as $n\to\infty$, so that

$$K_n/U_n\xrightarrow{\text{d}}\mathrm{N}(0,1) \quad \text{and} \quad K_n/V_n\xrightarrow{\text{d}}\mathrm{N}(0,1) \quad \text{as } n\to\infty.$$

Proof. To prove the theorem, we need to prove that, as $n\to\infty$

(i) $U_n/\sigma_n\xrightarrow{\text{p}}1$;

(ii) a strong asymptotic negligibility condition, for example, $\mathbf{E}\{\max_j|\Delta K_{nj}|^2/\sigma_n^2\}\to 0$: see [2].

It should be noted that these two conditions imply that $V_n/\sigma_n\xrightarrow{\text{p}}1$ since, as shown by McLeish, under the strong asymptotic negligibility condition, $U_n/\sigma_n\xrightarrow{\text{p}}1\Rightarrow V_n/\sigma_n\xrightarrow{\text{p}}1$.

Now, we have

$$U_n^2=\nabla H_2(X_n)+n^{-2}\theta^2 Z_n,$$

and, if $n^{-1}N_n\xrightarrow{\text{p}}\tau$, where $\tau>0$, then it follows from Lemma 1 that, as $n\to\infty$

$$nU_n^2\xrightarrow{\text{p}}\psi \quad \text{and} \quad n\sigma_n^2=\mathbf{E}(nU_n^2)\to\psi,$$

where $\psi=u(\tau)^{-1}-1+\theta^2 w(\tau)$. Hence, $U_n/\sigma_n\xrightarrow{\text{p}}1$ as $n\to\infty$.

The martingale increments ΔK_{nj} are such that

$$\mathbf{P}\{\Delta K_{nj} = -X_{nj}^{-1} \,|\, F_{n,j-1}\} = \lambda_n X_{n,j-1}(1+\lambda_n X_{n,j-1})^{-1}$$

and

$$\mathbf{P}\{\Delta K_{nj} = \lambda_n \,|\, F_{n,j-1}\} = (1+\lambda_n X_{n,j-1})^{-1}.$$

Therefore, $M_n = \max_j |\Delta K_{nj}| = \max\{(X_n+1)^{-1},\ n^{-1}\theta\}$, and so, from Lemma 2 we have

$$n\mathbf{E}(M_n^2) \to 0 \quad \text{as } n \to \infty.$$

Thus,

$$\mathbf{E}(M_n^2)/\sigma_n^2 \to 0 \quad \text{as } n \to \infty.$$

Hence the result.

Theorem 2. *For the sequence of processes $\mathcal{P}_n$, $n = 1, 2, \ldots$, defined above, if $a_n = a$, where $a > 0$, and $\lambda_n = n^{-1}\theta$, where $\theta > 1$, then*

$$K_n^+/\sigma_n^+ \xrightarrow{\mathrm{d}} \mathrm{N}(0, 1)$$

provided that $n^{-1}N_n^+ \le \nu$ and $n^{-1}N_n^+ \xrightarrow{\mathrm{p}} \tau$, where ν and τ are positive constants. The superscript $^+$ indicates conditioning on the event $E_n^+ = \{Y_{n,k_n} \ge \frac{1}{2}(\theta-1)k_n\}$, where $k_n = [\ln n]$. Further, $U_n^+/\sigma_n^+ \xrightarrow{\mathrm{p}} 1$ and $V_n^+/\sigma_n^+ \xrightarrow{\mathrm{p}} 1$ as $n \to \infty$, so that

$$K_n^+/U_n^+ \xrightarrow{\mathrm{d}} \mathrm{N}(0, 1) \quad \text{and} \quad K_n^+/V_n^+ \xrightarrow{\mathrm{d}} \mathrm{N}(0, 1) \quad \text{as } n \to \infty.$$

Proof. The process $\{Y_{nj}, j = 0, 1, 2, \ldots, k_n\}$ is stochastically bounded above and below by simple random walks with upward step probabilities $\theta(1+\theta)^{-1}$ and $\theta(1-n^{-1}k_n)[1+\theta(1-n^{-1}k_n)]^{-1}$. Thus, if $n^{-1}k_n < 1-\theta^{-1}$, then the lower bounding process is supercritical and we see that, as $n \to \infty$

$$\mathbf{P}(E_n^+) = \mathbf{P}(Y_{n,k_n} \ge \tfrac{1}{2}(\theta-1)k_n) \to 1-\theta^{-a}.$$

The event E_n^+ is thus asymptotically equivalent to the explosion of the bounding random walks and hence to a major outbreak in the epidemic model.

We define the process $\mathcal{P}_n^* = \{\boldsymbol{P}_{nj}^*, j = 0, 1, 2, \ldots\}$, where

$$\boldsymbol{P}_{nj}^* = \boldsymbol{P}_{n,k_n+j}^+ - \boldsymbol{P}_{n,k_n}^+, \quad j = 0, 1, 2, \ldots.$$

Now, if $Y_{n,k_n} \ge \frac{1}{2}(\theta-1)k_n$, then since $E_n^+ \in F_{n,k_n}$, it follows that $\mathcal{P}_n^*$ is a process to which Theorem 1 can be applied. Thus we have, as $n \to \infty$,

$$K_n^*/\sigma_n^* \xrightarrow{\mathrm{d}} \mathrm{N}(0, 1), \qquad K_n^*/U_n^* \xrightarrow{\mathrm{d}} \mathrm{N}(0, 1), \qquad K_n^*/V_n^* \xrightarrow{\mathrm{d}} \mathrm{N}(0, 1).$$

Further, we have

$$K_n^+ = K_{n,k_n}^+ + K_n^*, \qquad U_{n,k_n^2}^+ + U_n^{*2},$$

$$V_n^{+2} = V_{n,k_n^2}^+ + V_n^{*2}, \qquad \sigma_n^{+2} = \sigma_{n,k_n^2}^+ + \sigma_n^{*2};$$

and in each case the contribution of the first term, arising from the first k_n transitions, is asymptotically negligible, since for some positive constant c,

$$|K_{n,k_n}^+| < cn^{-1}k_n, \quad nU_{n,k_n^2}^+ < cn^{-1}k_n,$$

$$nV_{n,k_n^2}^+ < cn^{-1}k_n, \quad n\sigma_{n,k_n^2}^+ < cn^{-1}k_n.$$

Thus we have, as $n \to \infty$

$$K_n^+/\sigma_n^+ \xrightarrow{\mathrm{d}} \mathrm{N}(0,1), \qquad K_n^+/U_n^+ \xrightarrow{\mathrm{d}} \mathrm{N}(0,1), \qquad K_n^+/V_n^+ \xrightarrow{\mathrm{d}} \mathrm{N}(0,1).$$

We note that the event E_n^+ could equally well be defined as $\{Y_{n,t_n} \geq \lambda t_n\}$, where $t_n \to \infty$, $n^{-1}t_n \to 0$ as $n \to \infty$, and $0 < \lambda < \theta - 1$.

The remainder of the section is concerned with verifying that the theorems apply with stopping time $N_n = J_n(B_0)$.

Lemma 3. *Let S_n denote the size of the outbreak, so that $S_n = n - X_{n,J_n(B_0)}$, i.e., the reduction in the number of susceptibles when the number of infectives first reaches zero.*

(i) *If $a_n \to \infty$, $n^{-1}a_n \to \alpha$, where $\alpha \geq 0$, $\lambda_n = n^{-1}\theta$, where $\theta > 1$, then*

$$n^{-1}S_n \xrightarrow{\mathrm{p}} \sigma_\alpha \quad \text{as } n \to \infty,$$

where σ_α denotes the positive solution of the equation

$$\ln(1-\sigma) + \theta(\sigma + \alpha) = 0.$$

(ii) *If $a_n = a$, where $a > 0$, $\lambda_n = n^{-1}\theta$, where $\theta > 1$, then*

$$n^{-1}S_n^+ \xrightarrow{\mathrm{p}} \sigma_0 \quad \text{as } n \to \infty,$$

and hence

$$n^{-1}S_n \xrightarrow{\mathrm{d}} W \quad \text{as } n \to \infty,$$

where $\mathbf{P}(W = \sigma_0) = \theta^{-a}$, $\mathbf{P}(W = 0) = 1 - \theta^{-}a$.

Proof. (i) We have

$$\mathbf{P}(S_n \geq n - nu) = \mathbf{P}\{Y_n(x) > 0, x = [nu] + 1, \ldots, n\},$$

where $Y_n(x)$ denotes the value of Y_{nj} when X_{nj} is first equal to x, i.e., $Y_n(x) = Y_{n,J_n(A_x)}$.

Now,

$$Y_n(x) = a_n + \sum_{\xi=x+1}^{n} j_n(\xi),$$

where $j_n(\xi)$ has a geometric distribution with parameter $p = \lambda_n\xi(1+\lambda_n\xi)^{-1}$, so that

$$\mathbf{E}(j_n(\xi)) = 1 + \lambda_n^{-1}\xi^{-1}, \qquad \operatorname{var}(j_n(\xi)) = \lambda_n^{-1}\xi^{-1} + \lambda_n^{-2}\xi^{-2}.$$

Let $j_n^*(\xi) = j_n(\xi) - \mathbf{E}(j_n(\xi))$, then

$$Y_n(x) = \mu_n(x) + \sum_{\xi=x+1}^{n} j_n^*(\xi),$$

where $\mu_n(x) = \mathbf{E}(Y_n(x)) = a_n + n - x + \lambda_n^{-1}\nabla H_1(x)$. Now, using Kolmogorov's inequality we deduce that, as $n \to \infty$

$$\mathbf{P}\left\{\left|\sum_{\xi=x+1}^{n} j_n^*(\xi)\right| < \delta_n, x = [nu]+1, \ldots, n\right\} \to 1,$$

where $\delta_n = cn^{3/4}$, hence

$$\mathbf{P}\{\mu_n(x) - \delta_n < Y_n(x) < \mu_n(x) + \delta_n, x = [nu]+1, \ldots, n\} \to 1.$$

Now, as $n \to \infty$

$$\mu_n(x) \sim n(\alpha + 1 - n^{-1}x - \theta^{-1}\ln(n^{-1}x)),$$

so for sufficiently large n

$$\mu_n(x) - \delta_n > 0 \quad \text{for all } x > n(1-\sigma_\alpha),$$

$$\mu_n(x) + \delta_n < 0 \quad \text{for all } x < n(1-\sigma_\alpha).$$

Therefore,

$$\text{if } u > 1-\sigma_\alpha, \text{ then } \mathbf{P}\{Y_n(x) > 0, x = [nu]+1, \ldots, n\} \to 1,$$

$$\text{if } u < 1-\sigma_\alpha, \text{ then } \mathbf{P}\{Y_n(x) > 0, x = [nu]+1, \ldots, n\} \to 0.$$

Hence

$$\mathbf{P}(S_n < nr) \to \begin{cases} 0, & \text{if } r < \sigma_\alpha, \\ 1, & \text{if } r > \sigma_\alpha, \end{cases}$$

so that

$$n^{-1}S_n \xrightarrow{\mathrm{P}} \sigma_\alpha \quad \text{as } n \to \infty.$$

(ii) Using the method of Theorem 2, i.e., considering the state of the process after k_n transitions, we see that if E_n^+ occurs, then $\mathscr{P}_n^*$ is such that $a_n^* \geq \frac{1}{2}(\theta-1)k_n$ so that the conditions of (i) are satisfied by $\mathscr{P}_n^*$. It follows that if E_n^+ occurs then $n^{-1}S_n \xrightarrow{\mathrm{P}} \sigma_0$.

On the other hand, the event $\bar{E}_n^+$ is asymptotically equivalent to the early extinction of the process, so that if $\bar{E}_n^+$ occurs then $n^{-1}S_n \xrightarrow{\text{P}} 0$. Thus, since $\mathbf{P}(E_n^+) \to \theta^{-a}$ as $n \to \infty$, the result follows.

Lemma 4. (i) *If $a_n \to \infty$, $n^{-1}a_n \to \alpha$, where $\alpha \geq 0$, as $n \to \infty$, and $\lambda_n = n^{-1}\theta$, where $\theta > 1$, then $n^{-1}J_n(B_0) \xrightarrow{\text{P}} \tau_\alpha$ as $n \to \infty$, where $\tau_\alpha = 2\sigma_\alpha + \alpha$.*

(ii) *If $a_n = a$, where $a > 0$, and $\lambda_n = n^{-1}\theta$, where $\theta > 1$, then $n^{-1}J_n^+(B_0) \xrightarrow{\text{P}} \tau_0$ as $n \to \infty$.*

Proof. The result follows immediately from Lemma 3 on observing that $J_n(B_0) = 2S_n + a_n$.

Acknowledgment

I am grateful to Peter Hall for a number of helpful comments.

References

[1] A.D. Barbour, On a functional central limit theorem for Markov population processes, Advances in Appl. Probability 6 (1974) 21–39.
[2] D.L. McLeish, Dependent central limit theorems and invariance principles, Ann. Probability 2 (1974) 620–628,
[3] A.V. Nagaev and A.V. Startsev, The asymptotic analysis of a stochastic model of an epidemic, Theor. Probability Appl. 15 (1970) 98–107.
[4] R.K. Watson, Stochastic population models, Ph.D. Thesis, University of Melbourne (1977).
[5] R.K. Watson, A useful random time scale transformation for the standard epidemic model, J. Appl. Probability 17 (1980) 324–332.